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B.A./B.SC. THIRD SEMESTER EXAMINATION, DECEMBER 2012
SECOND YEAR
MATHEMATICS (Honours)
Paper : III

Date : 14/12/2012
Time : 11 am – 3 pm
Full Marks : 100

Use separate answer-book for each group

Group-A

Answer any five questions: 5x10

1. a) Define quotient space V/W, where V is a vectorspace over a field $(F,+,\cdot)$ & W is a subspace of V. If $T:V\to V^{/}$ be a linear transformation of V onto $V^{/}$ & W=Ker T then prove that V/W is isomorphic to $V^{/}$. 1+4

   b) $A=\begin{pmatrix} 1 & 2 & 5 \\ 3 & 0 & 7 \\ -1 & 4 & 3 \end{pmatrix}$. Find a basis of the row space of A. Examine if (1,-1,1) is in the row space of A. 5

2. a) Determine the conditions for which the system of equations

$$x+2y+z=1$$
$$2x+y+3z=b$$
$$x+ay+3z=b+1$$

   have (i) only one solution
   (ii) No solution
   (iii) Many solutions. 5

   b) A linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ is defined by T(x, y,z)=(x+y+z, y+z, z). Show that T is an isomorphism and determine T^{-1}. 2

   c) Use Cayley-Hamilton theorem to find A^{100} where $A=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. 3

3. a) If S be a skew hermitian matrix then prove that its eigen values are either zero or purely imaginary. Hence show that if λ is an eigen value of S then $\left|\frac{1-\lambda}{1+\lambda}\right|=1$. 3+2

   b) Prove that an orthogonal set of non null vectors in an inner product space V is linearly independent. 3

   c) Prove that eigen values of a Hermitian matrix are all real. 2

4. a) If $\{\beta_1,\beta_2,\cdots\beta_r\}$ be an orthonormal set of vectors in a Enclidean space V then for any vector α in V show that $\|\alpha\|^2\geq c_1^2+c_2^2+\cdots+c_r^2$ where c_i is the scalar component of α along β_i . When will the equality occur? (i=1,2,3,....r). 4+2

   b) Apply Gram-Schmidt process to find an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product, spanned by the vectors (1,1,1,1), (1,1,-1,-1), (1,2,0,2). 4

5. a) The matrix of a linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3$ w.r.t the ordered basis (0,1,1), (1,0,1) & (1,1,0) of $\mathbb{R}^3$ is given by $\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$

   Find the T & also find the matrix of T relative to the ordered basis {(2,1,1), (1,2,1) & (1,1,2)} 2+3

b) Find the orthogonal complement of the row space of the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix}$. 5

6. a) Diagonalise the matrix $A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$. 5

b) Show that the quadratic form $x^2 + 5y^2 + 2z^2 - 4xy - 6yz + 2zx$ is positive semi-definite. 5

7. a) Define Adjoint of a linear operator on an inner product space. For any linear operator T on a finite-demensional inner product space V prove that there exists a unique adjoint linear operator T^{*}. 1+5

b) Let V and W be two finite demensional vector spaces over the same field F. Prove that V and W are isomorphic if and only if dim V=dim W. 4

8. a) The matrix of $T:\mathbb{R}^2 \to \mathbb{R}^2$ is given by $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with respect to the ordered basis {(1,0), (0,1)} of $\mathbb{R}^2$. Find T and T* and verify whether T is a normal operator, here T* represents the adjoint of T. 4

b) State spectral theorem for normal operators on an inner product space. 2

c) Define uninary operator on an inner product space. If T is a linear operator on an inner product space V, prove that T is unitary iff the adjoint T* of T exists & $T^* = T^{-1}$. 1+3

<u>Group – B</u>

Answer <u>any four</u> questions from question no. 9 to question no. 14 & answer <u>any two</u> questions from question no. 15 to question no. 17

9. A variable line always intersects the line $z = 0,\ x = y$ and the circles $y^2+z^2=d^2$, x=0; $z^2+x^2=d^2$, y=0. Show that the equation to its locus is $(x+y)^2[z^2+(x-y)^2]=d^2(x-y)^2$. 5

10. Assuming the plane 4x-3y+7z=0 to be horizontal, show that the equations of the line of greatest slope through the point (2,1,1) in the plane 2x+y-5z = 0 are $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$. 5

11. The section of a cone whose guiding curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ by the plane x=0 is a rectangular hyperbola. Show that the locus of the vertex is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2} = 1$. 5

12. The line 2x=y=2z represents one line of the set of three mutually perpendicular generators of the cone 11yz+6zx-14xy=0. Find the equations of the other two generators. 5

13. Prove that among all central conicoids, the hyperboloid of one sheet is only the ruled. 5

14. Reduce the equation $3x^2+5y^2+3z^2-2yz+2zx-2xy+2x+12y+10z+20=0$ to its Canonical form and state the nature of the surface represented by it. 4+1

15. a) Two bodies M and M' are attached to the lower end of an elastic string whose upper end is fixed and are hung at rest, M' falls off. Show that the distance of M from the upper end of the string at time t is $a + b + c\ cos\sqrt{\frac{g}{b}}$ t, where a is the unstretched length of the string, b and c are distances by which it would be extended when supporting M and M' respectively. 6

b) If a particle moves under a central force in a medium which exerts a resistance equal to k times the velocity per unit of mass, then prove that the differential equation of its path can be written in the form $\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}.e^{2kt}$,

where h is twice the initial moment of momentum about the centre of force and the other symbols have their usual meanings. 6

c) Prove that every planet moves in a fixed plane through the centre of its sun. 3

16. a) Find the tangential and normal components of velocity and acceleration of a particle describes a plane curve. 7

b) A particle moves in a plane under a force, towards a fixed centre, proportional to the distance. If the path of the particle has two apsidal distances a, b (a>b), show that its equation can be written in the form $u^2=\frac{cos^2\theta}{a^2}+\frac{sin^2\theta}{b^2}$. 8

17. a) A particle of mass m is falling under the influence of gravity through a medium whose resistance is equal to μ times the velocity. If the particle is released from rest, show that the distance fallen through in time t is $g\frac{m^2}{\mu^2}\{e^{-\frac{\mu t}{m}}-1+\frac{\mu t}{m}\}$. 8

b) Two particles of masses M and m, are connected by a light string which passes through a small hole in a table. The latter hangs vertically and the former describes a curve on the table which is very nearly a circle whose centre is the hole. Show that the apsidal angle of the orbit of M is $\pi\sqrt{\frac{M+m}{3M}}$. 7